Continued fractions and combinatorial sequences: Factorial, Genocchi and median Genocchi numbers

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Based on Ongoing Joint Work With
Alan D. Sokal

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$$\underbrace{\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}}_{\text{Stieltjes continued fraction}} = \sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes-Rogers polynomials}} t^n$$

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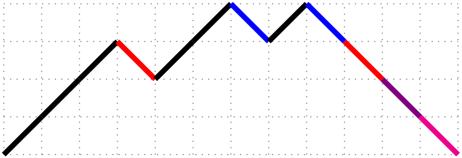

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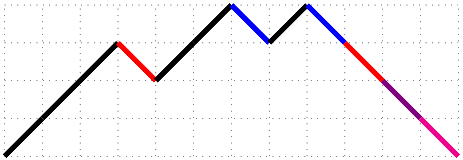

Assign weights:

- $\nearrow$ from height $(i-1) \to i$ — β_i
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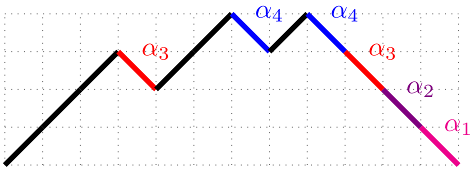

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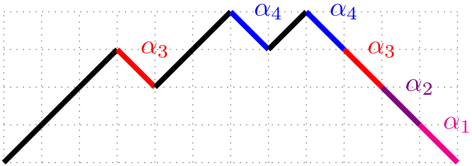

$$\text{Weight} = \alpha_1\beta_1\alpha_2\beta_2\alpha_3^2\beta_3^2\alpha_4^2\beta_4^2$$

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Theorem (Flajolet 1980)

Stieltjes-Rogers polynomial $S_n(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the weighted sum over all Dyck paths of semilength n.

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Also, Jacobi-Rogers polynomials ↔ Motzkin paths

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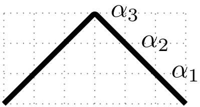

$\alpha_1\alpha_2\alpha_3$

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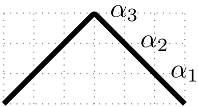

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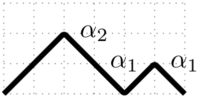

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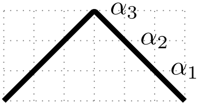

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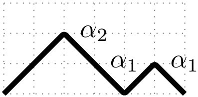

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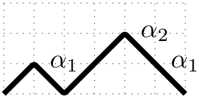

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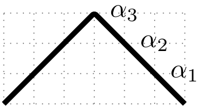

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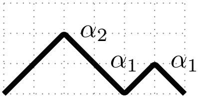

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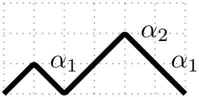

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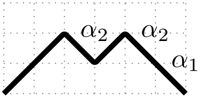

$\alpha_1\alpha_2^2$

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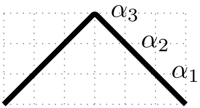

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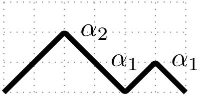

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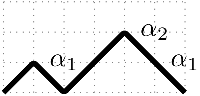

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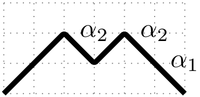

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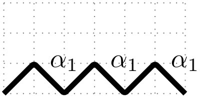

α_1^3

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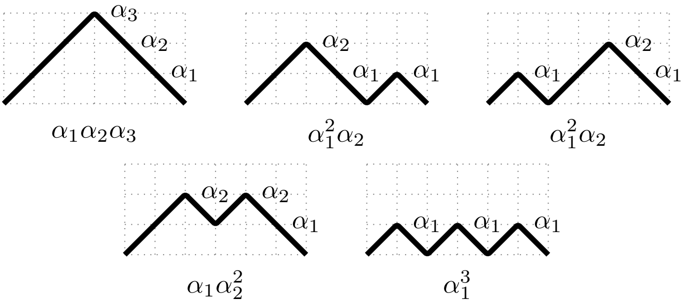

Weighted sum:

$$\alpha_1\alpha_2\alpha_3 + 2\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 + \alpha_1^3 = S_3(\boldsymbol{\alpha})$$

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- $(2n-1)!!$ (number of involutions without fixed points):
  α's are $1, 2, 3, 4, 5, \ldots$

A continued fraction due to Euler (1760)

Theorem

$$1 + 1!t + 2!t^2 + 3!t^3 + 4!t^4 + \ldots = \cfrac{1}{1 - \cfrac{1 \cdot t}{1 - \cfrac{1 \cdot t}{1 - \cfrac{2 \cdot t}{1 - \cfrac{2 \cdot t}{1 - \cfrac{3 \cdot t}{1 - \cfrac{3 \cdot t}{1 - \ddots}}}}}}}$$

The sequence of α*'s are* $1, 1, 2, 2, 3, 3, \ldots$.

Several combinatorial proofs are known

- Françon-Viennot bijection 1979
- Foata-Zeilberger bijection 1990
- Biane bijection 1993

Another continued fraction due to Euler (1760)

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In fact,

$$1 + xt + x(x+1)t^2 + x(x+1)(x+2)t^3 + \ldots = \cfrac{1}{1 - \cfrac{x \cdot t}{1 - \cfrac{1 \cdot t}{1 - \cfrac{(x+1) \cdot t}{1 - \cfrac{2 \cdot t}{1 - \cfrac{(x+2) \cdot t}{1 - \cfrac{3 \cdot t}{1 - \ddots}}}}}}}$$

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Define $P_n(x, y, u, v)$ such that

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Question: What permutation statistics do y,u,v count?

Specialisations of $P_n(x, y, u, v)$

- Catalan numbers (number of Dyck paths):
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Theorem (Sokal-Zeng 2022)

(a)

$$P_n(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{arec}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)}$$

(b)

$$P_n(x,y,u,v) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{erec}(\sigma)} u^{n-\mathrm{exc}(\sigma)-\mathrm{cyc}(\sigma)} v^{\mathrm{exc}(\sigma)-\mathrm{erec}(\sigma)}$$

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- arec - antirecords or right-to-left minima
- rec - records or left-to-right maxima
- erec - exclusive records i.e. records that are not anti-records
- exc - excedances i.e. $(i, \sigma(i))$ such that $i < \sigma(i)$

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Randrianarivony in a little-known paper had actually interpreted almost all of the variables for different statistics in 1998!!!

Question

We have a combinatorial interpretation for

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i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.

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- $k=1$ quasi-linear case: $n!$
- $k=2$ quasi-quadratic case: Median Genocchi numbers
- $k=3$ quasi-cubic case: Not on OEIS!!!

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Some combinatorial sequences whose α grow quadratically

- Genocchi numbers A11050
- Median Genocchi numbers A00543
- Once shifted median Genocchi numbers A00036
- Tangent numbers A00018
- Secant numbers A00036
- Even Springer numbers A00028

Genocchi numbers

The Genocchi numbers are given by

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The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$.

Combinatorial Interpretation

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

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D-e-semiderangement
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D-derangements
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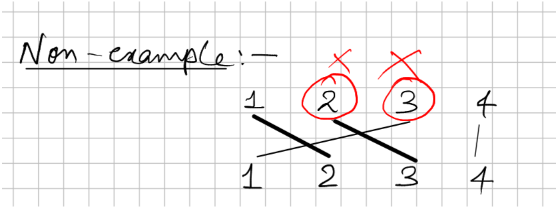
D-permutations or Dumont-like permutations (introduced by Lazar and Wachs 2019)

Non-example

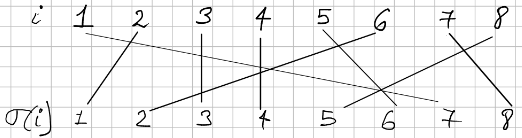


Example of a D-permutation

Example:-

D-permutations

$2i \geq \sigma(2i)$ $\quad$ $2i-1 \leq \sigma(2i-1)$



$\sigma = 713\ 46285$

Continued fractions

The g_n have an S-fraction with α's

$$1 \cdot 1, 1 \cdot 2, 2 \cdot 2, 2 \cdot 3, 3 \cdot 3, 3 \cdot 4, \ldots$$

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$$1, 1, 4, 4, 9, 9 \ldots$$

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and the h_{n+1} have an S-fraction with α's

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Want a unifying continued fraction for all three sequences.

Classical S-fractions with integer α due to Viennot (1981)

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Pan-Zeng (2021) have a multivariate continued fraction in 8 variables with linear statistics for a different combinatorial interpretation (even-odd descent permutations).

Main Theorem

We have a T-fraction (Thron-type continued fractions):
Constructed by using

- excedences ($(i, \sigma(i))$ with $i < \sigma(i)$) with parities (e or o)
- anti-excedances with parities
- fixed points with parities
- records, anti-records

$$P_n(x_{\text{ee}}, x_{\text{eo}}, u_{\text{ee}}, u_{\text{eo}}, y_{\text{oo}}, y_{\text{oe}}, v_{\text{oo}}, v_{\text{oe}}, z_{\text{o}}, z_{\text{e}}, w_{\text{o}}, w_{\text{e}}) \;=$$

$$\sum_{\sigma \in \mathfrak{D}_{2n}} x_{\text{ee}}^{\text{earecaexcee}(\sigma)} x_{\text{eo}}^{\text{earecaexceo}(\sigma)} u_{\text{ee}}^{\text{nraexcee}(\sigma)} u_{\text{eo}}^{\text{nraexceo}(\sigma)} \;\times$$

$$y_{\text{oo}}^{\text{erecexcoo}(\sigma)} y_{\text{oe}}^{\text{erecexcoe}(\sigma)} v_{\text{oo}}^{\text{nrexcoo}(\sigma)} v_{\text{oe}}^{\text{nrexcoe}(\sigma)} \;\times$$

$$z_{\text{o}}^{\text{raro}(\sigma)} z_{\text{e}}^{\text{rare}(\sigma)} w_{\text{o}}^{\text{nrfixo}(\sigma)} w_{\text{e}}^{\text{nrfixe}(\sigma)} \;.$$

$$P_n(x_{ee}, x_{eo}, u_{ee}, u_{eo}, y_{oo}, y_{oe}, v_{oo}, v_{oe}, z_o, z_e, w_o, w_e) =$$

$$\sum_{\sigma \in \mathfrak{D}_{2n}} x_{ee}^{\mathrm{earecaexcee}(\sigma)} x_{eo}^{\mathrm{earecaexceo}(\sigma)} u_{ee}^{\mathrm{nraexcee}(\sigma)} u_{eo}^{\mathrm{nraexceo}(\sigma)} \times$$

$$y_{oo}^{\mathrm{erecexcoo}(\sigma)} y_{oe}^{\mathrm{erecexcoe}(\sigma)} v_{oo}^{\mathrm{nrexcoo}(\sigma)} v_{oe}^{\mathrm{nrexcoe}(\sigma)} \times$$

$$z_o^{\mathrm{raro}(\sigma)} z_e^{\mathrm{rare}(\sigma)} w_o^{\mathrm{nrfixo}(\sigma)} w_e^{\mathrm{nrfixe}(\sigma)} .$$

$$\cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cfrac{\alpha_4 t}{1 - \cfrac{\alpha_5 t}{\ddots}}}}}}$$

where

$$\delta_1 = z_e z_o$$

$$\alpha_{2k-1} = [x_{eo} + (k-1)u_{eo}] \cdot [y_{oe} + (k-1)v_{oe}]$$

$$\alpha_{2k} = [x_{ee} + (k-1)u_{ee} + w_e] \cdot [y_{oo} + (k-1)v_{oo} + w_o].$$

General form of a T-fraction

$$\cfrac{1}{1-\delta_0 t-\cfrac{\alpha_1 t}{1-\delta_1 t-\cfrac{\alpha_2 t}{1-\delta_2 t-\cfrac{\alpha_3 t}{1-\delta_3 t-\cfrac{\alpha_4 t}{1-\delta_4 t-\cfrac{\alpha_5 t}{\ddots}}}}}}$$

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Schröder path

Schröder path - Path on non-negative quadrant

- starting at $(0, 0)$
- ending at $(2n, 0)$
- with steps $(1, 1)$, $(1, -1)$,

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Assign weights:

- step $(1, 1)$ $(\nearrow)$ from height $(i - 1) \to i$ — 1
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- step $(2, 0)$ from height $i \to i$ — δ_i

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Theorem (Elvey Price - Sokal 2020)

Thron-Rogers polynomial $T_n(\boldsymbol{\alpha},\boldsymbol{\delta})$ is the weighted sum over all Schröder paths of semilength n.

Bijection from D-permutations to labelled Schröder paths

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- Assign choice of labels

Description of Step 1

Let σ be a D-permutation on $2n$ letters.

- If $\sigma^{-1}(i)$ is even, step i is $\nearrow$
- If $\sigma^{-1}(i)$ is odd, step i is $\searrow$

Thank you