Combinatorics and Total Positivity

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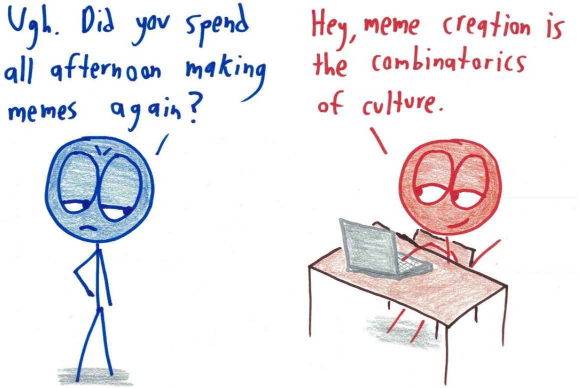
August 23, 2022
Chennai Mathematical Institute



Source: Math with Bad Drawings

Structure

1. Introduction
2. Proof techniques and some special types of matrices
    1. LGV lemma
    2. Hankel matrices
    3. Toeplitz matrices
    4. Lower triangular matrices
3. The Eulerian triangle

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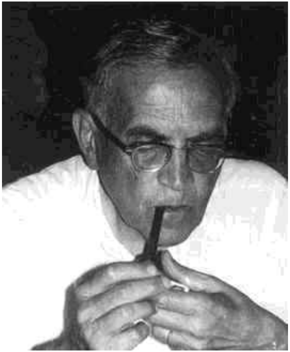
Will consider a matrix of polynomials soon!

Historical Note

First defined independently by two different groups in the 30s

(a) M.G. Krein (1907-1989)



(b) I.J. Schoenberg (1903-1990)

Source: MacTutor History of Mathematics Archive

We use Schoenberg's terminology.

Example

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- Example: Bidiagonal matrices with entries ≥ 0

$$\begin{bmatrix} 0 & a_1 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & b_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_5 & b_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

Some Operations which Preserve TP

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  where both A and B are TP.
- Matrix Products preserve TP.
  Proof: Use Cauchy-Binet formula which we recall here:

Theorem (Cauchy-Binet formula)

Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then we have that

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$$

where $[n]$ denotes the set $\{1, \ldots, n\}$, and given a set T, $\binom{T}{k}$ is the collection of all k-dimensional subsets of T.

Matrix Addition does not preserve TP

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

where

$$\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) = -1$$

Are there any more?

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Removes the necessity of checking non-negativity of all minors.

Coefficientwise TP

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But much stronger.

Structure

LGV lemma

The essence of mathematics is proving theorems - and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside-Frobenius Lemma in combinatorics.

. . .

- Chapter 25, Lattice Paths and Determinants
Proofs from the Book

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  - Sources – clockwise direction

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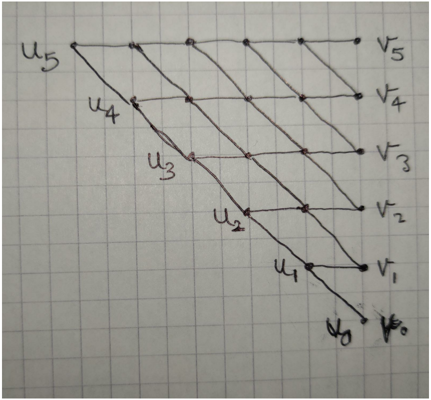

Example. Edges weighted 1.

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- Weight of a path – product of edges

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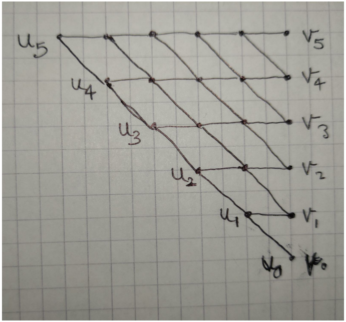
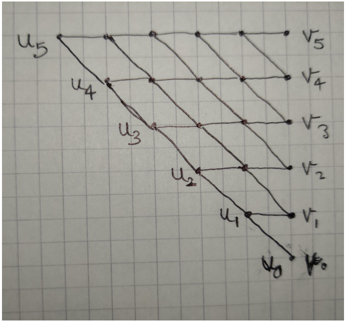
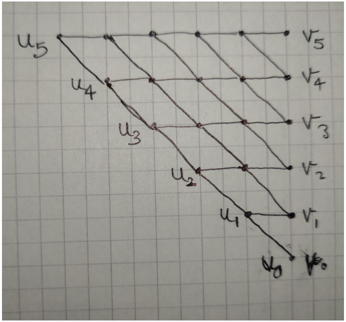

u5
u4
u3
u2
u1
v5
v4
v3
v2
v1



Path matrix

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$



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Binomial triangle!

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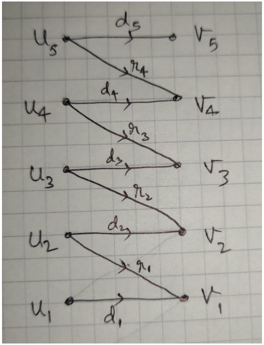
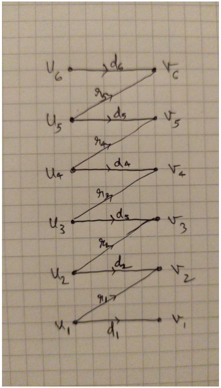
- Weight of a path – product of edges
- Path matrix
  - Entry (n, k) – sum of paths from u_n to v_k
- LGV lemma says that the path matrix is totally positive

Example: Lower Bidiagonal matrix



$$\begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ r_1 & d_2 & 0 & 0 & 0 \\ 0 & r_2 & d_3 & 0 & 0 \\ 0 & 0 & r_3 & d_4 & 0 \\ 0 & 0 & 0 & r_4 & d_5 \end{bmatrix}$$

Example: Upper Bidiagonal matrix



$$\begin{bmatrix} d_1 & r_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & r_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & r_3 & 0 & 0 \\ 0 & 0 & 0 & d_4 & r_4 & 0 \\ 0 & 0 & 0 & 0 & d_5 & r_5 \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{bmatrix}$$

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Converse not true in general. Not even true for integers as Neville factorisation involves division.

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Hankel Matrix

Given a sequence $a_0, a_1, \ldots$ the infinite matrix $H_\infty(\mathbf{a})$ whose ij^{th} entry is a_{i+j} is called the Hankel matrix of $(a_n)_{n \geq 0}$.

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\ a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\ a_3 & a_4 & a_5 & a_6 & a_7 & \ldots \\ a_4 & a_5 & a_6 & a_7 & a_8 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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a_3 & a_4 & a_5 & a_6 & a_7 & \dots \\
a_4 & a_5 & a_6 & a_7 & a_8 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}
$$

We say that a sequence $(a_n)_{n\geq 0}$ is Hankel-totally positive (Hankel-TP in short) if its Hankel matrix is TP.

Fundamental Fact about Hankel-TP

Theorem (Stieltjes(1894), Gantmacher-Krein(1937))

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$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}.$$

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For coefficientwise Hankel-TP, (3) $\Longrightarrow$ (1).

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$$\underbrace{\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}}}_{\text{Stieltjes continued fraction}} = \sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes-Rogers polynomials}} t^n$$

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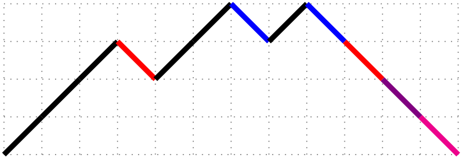

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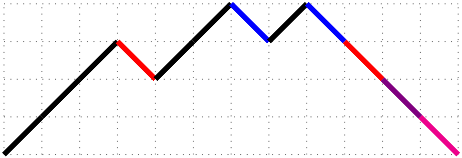

Assign weights:

- $\nearrow$ from height $(i-1) \to i$ — β_i
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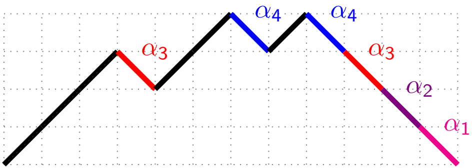

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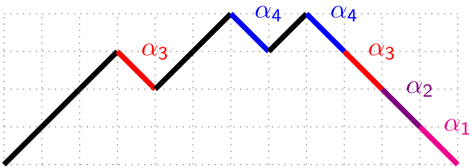

$$\text{Weight} = \alpha_1\beta_1\alpha_2\beta_2\alpha_3^2\beta_3^2\alpha_4^2\beta_4^2$$

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Theorem (Flajolet 1980)

Stieltjes-Rogers polynomial $S_n(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the weighted sum over all Dyck paths of semilength n.

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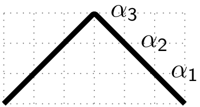
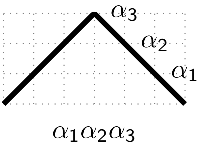
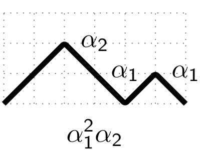


$\alpha_1\alpha_2\alpha_3$

$n = 3$

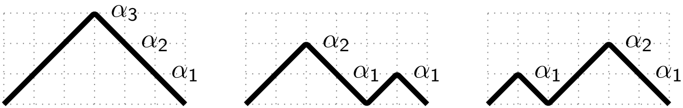


α_3
α_2
α_1
$\alpha_1\alpha_2\alpha_3$



α_2
α_1
α_1
$\alpha_1^2\alpha_2$

$n = 3$



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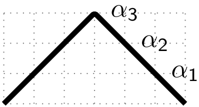

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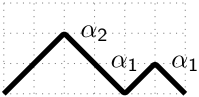

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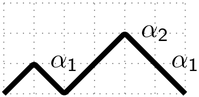

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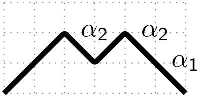

$\alpha_1\alpha_2^2$

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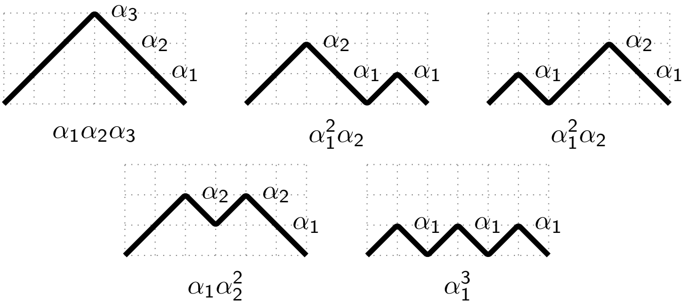
α_3
α_2
α_1
$\alpha_1\alpha_2\alpha_3$
α_2
α_1
α_1
$\alpha_1^2\alpha_2$
α_1
α_2
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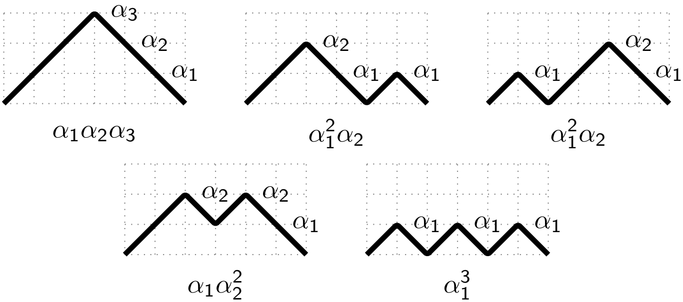
α_1^3

$n = 3$



Weighted sum:

$$\alpha_1\alpha_2\alpha_3 + 2\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 + \alpha_1^3 = S_3(\boldsymbol{\alpha})$$

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Sokal and his collaborators have developed several other continued fractions and associated path models which are other sufficient conditions for proving Hankel-TP.

Structure

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2. Proof techniques and some special types of matrices
   1. LGV lemma
   2. Hankel matrices
   3. Toeplitz matrices
   4. Lower triangular matrices
3. The Eulerian triangle

Toeplitz Matrix

Given a sequence $a_0, a_1, \ldots$ the infinite matrix $T_\infty(\mathbf{a})$ whose ij^{th} entry is a_{i-j} for $i \leq j$ and 0 otherwise is called the Toeplitz matrix of $(a_n)_{n \geq 0}$.

$$\begin{matrix} a_0 & 0 & 0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & 0 & 0 & \ldots \\ a_2 & a_1 & a_0 & 0 & 0 & \ldots \\ a_3 & a_2 & a_1 & a_0 & 0 & \ldots \\ a_4 & a_3 & a_2 & a_1 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

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We say that a sequence $(a_n)_{n\ge 0}$ is Toeplitz-totally positive (Toeplitz-TP in short). Also often called a Polya frequency sequence (PF sequence).

Fundamental Fact about Toeplitz-TP

Theorem (Aissen-Schoenberg-Whitney, Edrei 1952)

For a sequence $(a_n)_{n\geq 0}$ of real numbers with $a_0 = 1$, TFAE:

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For a finite sequence we need to show that the generating polynomial is negative real rooted.

Theorem (Katkova 2006)

Let $\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *be the Reimann-*ξ *function and let* $\xi_1(z) = \xi(\sqrt{z} + 1/2)$.

Equivalent statement of Riemann Hypothesis

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Consider the sequence $(a_n)_{n\geq 0}$.

Log Concavity and Log Convexity

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Toeplitz-TP_2 implies log-concavity i.e., $a_n^2 - a_{n-1}a_{n+1} \geq 0$.

Hankel-TP_2 implies log-convexity i.e., $a_n^2 - a_{n-1}a_{n+1} \leq 0$.

Structure

Lower Triangular Matrices and three types of TP

Consider an infinite lower triangular matrix A with entries a_{ij} where the indexing of the rows and columns begins from 0.

$$\begin{matrix} a_{00} & 0 & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & 0 & 0 & 0 & \dots \\ a_{20} & a_{21} & a_{22} & 0 & 0 & \dots \\ a_{30} & a_{31} & a_{32} & a_{33} & 0 & \dots \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Let $A_n(x)$ denote the row generating polynomial of the n^{th} row i.e.

$$A_n(x) = \sum_{i=0}^{n} a_{ni} x^i$$

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All three of these seem to be true for several important combinatorial triangles.

Example: Binomial Triangle

For A the triangle of binomial numbers,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

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3. The sequence $(A_n(x))$ is Hankel-TP as we have the easy continued fraction expansion

$$\sum_{n=0}^{\infty} (1+x)^n t^n = \frac{1}{1-(1+x)t}.$$

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Definition

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$$\left(\left\langle {n+1 \atop k} \right\rangle \right)_{n,k \geq 0} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ 1 & 57 & 302 & 302 & 57 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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General pattern not clear.

Stirling subset triangle

Entries $\left\{ {n+1 \atop k} \right\}$: set partitions of $\{1, \dots, n+1\}$ with k blocks

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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Brenti (1995) showed that this matrix is TP

Example 4: Reversed Stirling cycle triangle

Original Matrix

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 6 & 11 & 6 & 1 & & \\ 24 & 50 & 35 & 10 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Reversal

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 2 & & & \\ 1 & 6 & 11 & 6 & & \\ 1 & 10 & 35 & 50 & 24 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Reversed Stirling Subset Triangle

Reversed Stirling subset triangle:

$$\left(\left\{ {n+1 \atop k} \right\}^{\mathrm{rev}}\right)_{n,k\geq 0} = \left(\left\{ {n+1 \atop n-k+1} \right\}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Conjecture (Us 2019)

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A comparision of the two triangles

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$$A_{n,k} = {\color{magenta}(n-k+1)}A_{n-1,k-1} + {\color{magenta}1} \cdot A_{n-1,k}$$

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Putting variables in the recurrence

Replace

$$A_{n,k} = (n - k + 1)A_{n-1,k-1} + (k + 1)A_{n-1,k}$$

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Introduce variables a, c, d, e.

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Empirically True till 13×13

Special Cases

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$(1, 1, 1, 1)$	clean Eulerian triangle, conjecture
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$(0,1,1,1)$	Stirling subset numbers, TP proved by Brenti 1995
(1,1,0,1)	**Row reversed matrix of Stirling subset numbers, ~~conjecture~~**

Special Cases

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + (dk+e)A_{n-1,k}$$

Special cases:

(a,c,d,e)	Matrix obtained
$(1,1,1,1)$	clean Eulerian triangle, conjecture
$(1,0,1,1)$	shifted Eulerian triangle, conjecture
$(1,1,1,0)$	shifted Eulerian triangle, conjecture
$(0,1,0,1)$	Binomial triangle, TP
$(0,1,1,1)$	Stirling subset numbers, TP proved by Brenti 1995
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https://arxiv.org/pdf/2012.03629.pdf
FPSAC 2021

Reversed Stirling subset triangle

$$\left(\left\{ n+1 \atop k \right\}^{\text{rev}}\right)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 7 & 1 & & \\ 1 & 10 & 25 & 15 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

ace triangle

$$\begin{bmatrix} 1 \\ e & c \\ e^2 & ae+2ce & c^2 \\ e^3 & 3ae^2+3ce^2 & a^2e+3ace+3c^2e & c^3 \\ e^4 & 6ae^3+4ce^3 & 7a^2e^2+12ace^2+6c^2e^2 & a^3e+4a^2ce+6ac^2e+4c^3e & c^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

recurrence

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + eA_{n-1,k}$$

ace triangle

$$\begin{bmatrix} 1 \\ e & c \\ e^2 & ae+2ce & c^2 \\ e^3 & 3ae^2+3ce^2 & a^2e+3ace+3c^2e & c^3 \\ e^4 & 6ae^3+4ce^3 & 7a^2e^2+12ace^2+6c^2e^2 & a^3e+4a^2ce+6ac^2e+4c^3e & c^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

recurrence

$$A_{n,k} = [a(n-k)+c]A_{n-1,k-1} + eA_{n-1,k}$$

Also satisfies the alternate recurrence:

$$A_{n,k} = cA_{n-1,k-1} + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e A_{n-1-m,k-m}$$

ace triangle

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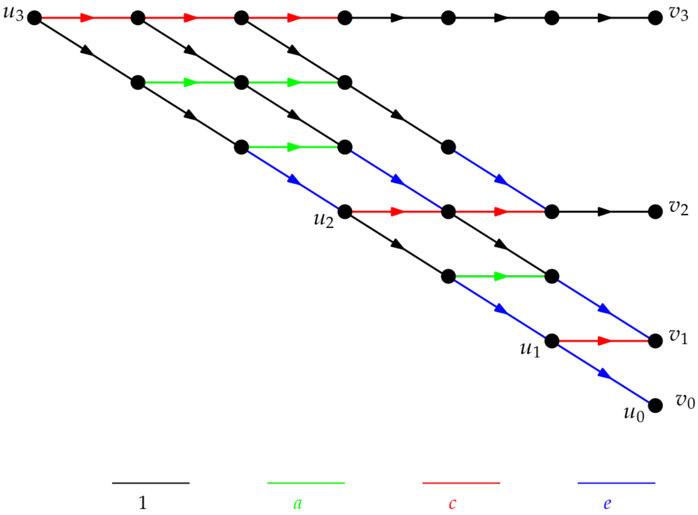
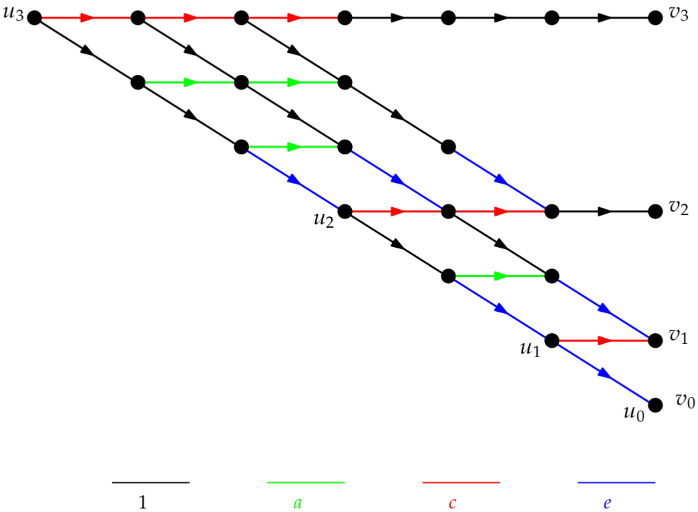
We have two and a half proofs:

- Two Digraph proofs using the Lindström-Gessel-Viennot lemma
  - Alternate recurrence
  - Direct bijection
- Algebraic proof

An *ace* digraph



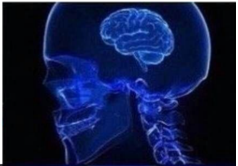
An *ace* digraph



We have a weight preserving bijection between paths from $u_n \to v_k$ and set partitions of $\{1, \ldots, n+1\}$ into $n - k + 1$ parts.

$$T(n,k) = (n-k+1)T(n-1,k-1) + (k+1)T(n-1,k)$$

Eulerian triangle conjecture



$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k)$$

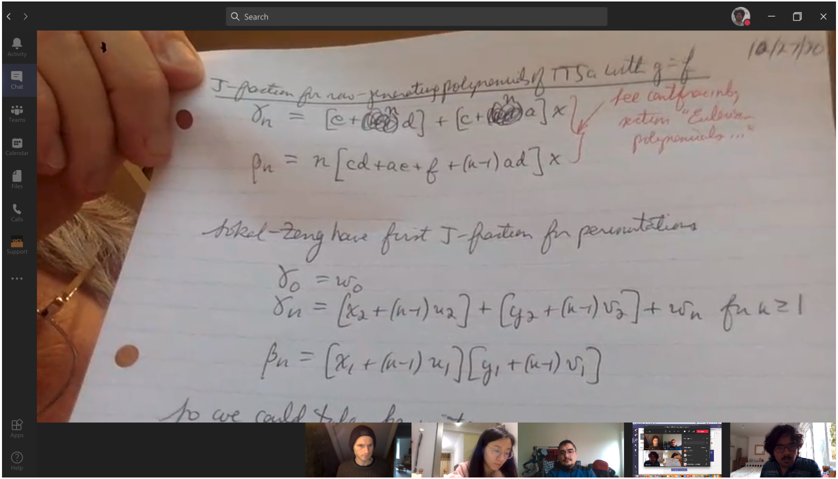
acde conjecture

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + (dk+e)T(n-1,k) + [f(n-2)+g]T(n-2,k-1)$$

acdefg conjecture

$$T(n,k) = [a(n-k)+c]T(n-1,k-1) + \left(\sum_{i=0}^{k} d_i\right) T(n-1,k) + [f(n-2)+g]T(n-2,k-1)$$

$ac\mathbf{d}fg$ conjecture



Thank you

https://arxiv.org/pdf/2012.03629.pdf

Meme images from internet.