The “quadratic family” of continued fractions and combinatorial sequences

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Topics in Special Functions and Number Theory

Based on Joint Work With

Alan D. Sokal

Structure

Structure

1. Introduction
2. Tangent, Secant, Genocchi, Genocchi medians
3. The permutations story
4. The D-permutations story
5. The cycle-alternating permutations story
6. Jacobi–Rogers matrix

Stieltjes-type continued fraction (S-fraction):

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Also called regular C-fraction outside of combinatorial literature.

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Classify sequences by growth of α

Linear α

- Catalan numbers:
  α's are $1, 1, 1, 1, \ldots$

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- Bell numbers (number of set partitions):
  α's are $1, 1, 1, 2, 1, 3, 1, 4 \ldots$

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- Bell numbers (number of set partitions):
  α's are $1, 1, 1, 2, 1, 3, 1, 4 \ldots$
- $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$:
  α's are $1, 2, 3, 4, 5, \ldots$

Quadratic α

- Tangent numbers A000182
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E_{2n+1} - Tangent numbers
α's are $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, 5 \cdot 6, \ldots$

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Classically expressed using Borel summation

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The first few numbers are $1, 1, 3, 17, 155, 2073, \ldots$

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α's are $1\cdot 1, 1\cdot 2,\ 2\cdot 2, 2\cdot 3,\ 3\cdot 3, 3\cdot 4\ldots$
(Viennot 1981)

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Combinatorial Interpretation for Genocchi and Genocchi medians

Genocchi numbers g_n are counted by

$$\#\{\sigma \in \mathfrak{S}_{2n} | 2i > \sigma(2i) \text{ and } 2i-1 \leq \sigma(2i-1)\}$$

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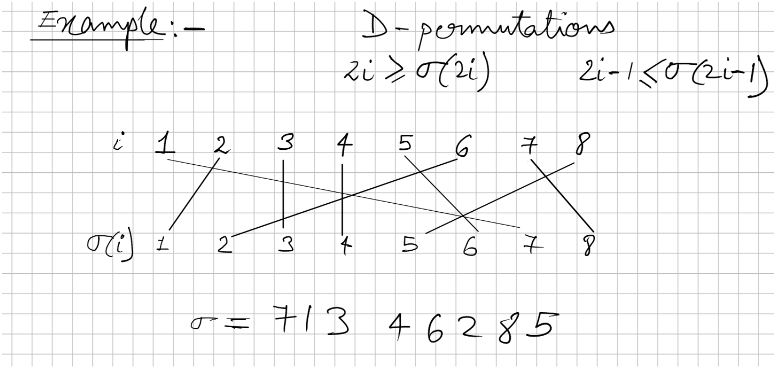
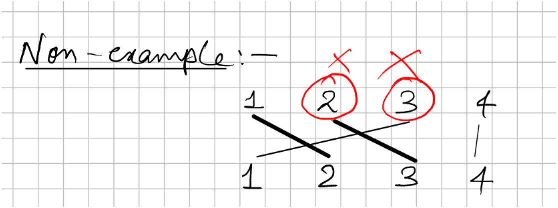
D-permutations or Dumont-like permutations (Lazar and Wachs 2019)

Example of a D-permutation



Non-example



Structure

Combinatorics and continued fractions: The permutations story

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Jacobi-type continued fraction for $n!$:

$$1 + 1!t + 2!t^2 + 3!t^3 + 4!t^4 + \ldots = \cfrac{1}{1 - 1 \cdot t - \cfrac{1 \cdot t^2}{1 - 3 \cdot t - \cfrac{4 \cdot t^2}{1 - 5 \cdot t - \cfrac{9 \cdot t^2}{1 - \ddots}}}}$$

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Also called associated C-fraction outside of combinatorial literature.

Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

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- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Consider 5-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, w) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{cpeak}(\sigma)} x_2^{\mathrm{cdfall}(\sigma)} y_1^{\mathrm{cval}(\sigma)} y_2^{\mathrm{cdrise}(\sigma)} z^{\mathrm{fix}(\sigma)}$$

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J-fraction:

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, w) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + z) \cdot t - \cfrac{4 x_1 y_1 \cdot t^2}{1 - (2x_2 + 2y_2 + z) \cdot t - \cfrac{9 x_1 y_1 \cdot t^2}{1 - \ddots}}}}$$

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\ldots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
- i is antirecord if for every $i > j$ we have $\sigma(i) < \sigma(j)$
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Each i is one of the following four types:

- rar - record-antirecord
- erec - exclusive record
- earec - exclusive antirecord
- nrar - neither record-antirecord

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Each i is one of the following ten (not 20) types:

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- nrcdfall
- rar
- nrfix

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$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

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Nice J-fraction:

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n = \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

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Randrianarivony in a little-known paper had actually interpreted almost all of the variables for different statistics in 1998!!!

Counting cycles

Consider 11-variable polynomials

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Theorem (D. (2023), Conjectured by Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda) t^n$$

$$= \cfrac{1}{1 - \lambda z \cdot t - \cfrac{\lambda x_1 \, y_1 \cdot t^2}{1 - (x_2 + y_2 + \lambda w) \cdot t - \cfrac{(\lambda+1)(x_1+u_1) y_1 \cdot t^2}{1 - ((x_2+v_2) + (y_2+v_2) + \lambda w) \cdot t - \cfrac{(\lambda+2)(x_1+2u_1) y_1 \cdot t^2}{1 - \ddots}}}}$$

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Consider 10-variable polynomial

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Thron-type continued fraction

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}$$

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where

$$\delta_1 = z^2$$

$$\alpha_{2k-1} = [x_1 + (k-1)u_1] \cdot [y_1 + (k-1)v_1]$$

$$\alpha_{2k} = [x_2 + (k-1)u_2 + w] \cdot [y_2 + (k-1)v_2 + w].$$

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Can do better!!

Separate fixed points by parity

Theorem (D.-Sokal '22 (arxiv))

$$\sum_{n=0}^{\infty} P_n t^n = \cfrac{1}{1 - \delta_0 t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{\ddots}}}}$$

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where

$$\delta_1 = z_\mathrm{e} z_\mathrm{o}$$

$$\alpha_{2k-1} = [x_1 + (k-1)u_1] \cdot [y_1 + (k-1)v_1]$$

$$\alpha_{2k} = [x_2 + (k-1)u_2 + w_\mathrm{e}] \cdot [y_2 + (k-1) + v_2 + w_\mathrm{o}].$$

D-e-semiderangements

$\subseteq$ $\subseteq$

$\underbrace{\text{D-derangements}}_{h_n}$ $\subseteq$ $\subseteq$ $\underbrace{\text{D-permutations}}_{h_{n+1}}$

$\underbrace{\text{D-o-semiderangements}}_{g_n}$

$$\underbrace{\text{D-derangements}}_{h_n} \subseteq \text{D-e-semiderangements} \subseteq \underbrace{\text{D-permutations}}_{h_{n+1}}$$

$$\underbrace{\text{D-derangements}}_{h_n} \subseteq \underbrace{\text{D-o-semiderangements}}_{g_n} \subseteq \underbrace{\text{D-permutations}}_{h_{n+1}}$$

We can also count cycles [D.–Sokal ’22, D. ’23]

$$\underbrace{\text{D-cycles}}_{g_{n-1}} \subseteq \underbrace{\text{D-derangements}}_{h_n} \;\begin{matrix} \subseteq \\ \subseteq \end{matrix}\; \begin{matrix} \text{D-e-semiderangements} \\ \\ \underbrace{\text{D-o-semiderangements}}_{g_n} \end{matrix} \;\begin{matrix} \subseteq \\ \subseteq \end{matrix}\; \underbrace{\text{D-permutations}}_{h_{n+1}}$$

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Structure

Combinatorial Interpretation for Secant numbers

Secant numbers E_{2n} are counted by

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Secant numbers E_{2n} are counted by cycle-alternating permutations

Combinatorial Interpretation for Secant numbers

Secant numbers E_{2n} are counted by cycle-alternating permutations $\sigma \in \mathfrak{S}_{2n}$ where each $i \in [2n]$

- either cycle valley $(\sigma^{-1}(i) > i < \sigma(i))$
- or cycle peak $(\sigma^{-1}(i) < i > \sigma(i))$

The cycle-alternating permutations story

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

Theorem (First J-fraction of Sokal–Zeng (2022) for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, z, w) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + v_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

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Set $x_2 = y_2 = u_2 = v_2 = w = z = 0$

Consider 4-variable polynomials

$$P_{2n}(x,y,u,v) = \sum_{\sigma\in\mathfrak{S}_{2n}^{\mathrm{ca}}} x^{\mathrm{eareccpeak}(\sigma)} y^{\mathrm{ereccval}(\sigma)} u^{\mathrm{nrcpeak}(\sigma)} v^{\mathrm{nrcval}(\sigma)}$$

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Theorem (First J-fraction of Sokal–Zeng (2022) for cycle-alternating permutations)

$$\sum_{n=0}^{\infty} P_{2n}(x,y,u,v)t^n = \cfrac{1}{1-\cfrac{x\,y\cdot t}{1-\cfrac{(x+u)\,(y+v)\cdot t}{1-\cfrac{(x+2u)\,(y+2v)\cdot t}{1-\ddots}}}}$$

The cycle-alternating permutations story

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Can do better

The cycle-alternating permutations story

Separate by parity

Separate by parity

$$
\begin{aligned}
& P_{2n}(x_{\rm e},x_{\rm o},y_{\rm e},y_{\rm o},u_{\rm e},u_{\rm o},v_{\rm e},v_{\rm o}) \\
= & \sum_{\sigma\in\mathfrak{S}_{2n}^{\rm ca}} x_{\rm e}^{\rm eareccpeakeven(\sigma)} x_{\rm o}^{\rm eareccpeakodd(\sigma)} y_{\rm e}^{\rm ereccvaleven(\sigma)} y_{\rm o}^{\rm ereccvalodd(\sigma)} \quad \times \\
& \qquad u_{\rm e}^{\rm nrcpeakeven(\sigma)} u_{\rm o}^{\rm nrcpeakodd(\sigma)} v_{\rm e}^{\rm nrcvaleven(\sigma)} v_{\rm o}^{\rm nrcvalodd(\sigma)}
\end{aligned}
$$

Separate by parity

$$
\begin{aligned}
& P_{2n}(x_\mathrm{e}, x_\mathrm{o}, y_\mathrm{e}, y_\mathrm{o}, u_\mathrm{e}, u_\mathrm{o}, v_\mathrm{e}, v_\mathrm{o}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x_\mathrm{e}^{\mathrm{eareccpeakeven}(\sigma)} x_\mathrm{o}^{\mathrm{eareccpeakodd}(\sigma)} y_\mathrm{e}^{\mathrm{ereccvaleven}(\sigma)} y_\mathrm{o}^{\mathrm{ereccvalodd}(\sigma)} \quad \times \\
& \quad u_\mathrm{e}^{\mathrm{nrcpeakeven}(\sigma)} u_\mathrm{o}^{\mathrm{nrcpeakodd}(\sigma)} v_\mathrm{e}^{\mathrm{nrcvaleven}(\sigma)} v_\mathrm{o}^{\mathrm{nrcvalodd}(\sigma)}
\end{aligned}
$$

Theorem (D.–Sokal '23 (arxiv))

$$
\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{x_\mathrm{e}\, y_\mathrm{o} \cdot t}{1 - \cfrac{(x_\mathrm{o} + u_\mathrm{o})\,(y_\mathrm{e} + v_\mathrm{e}) \cdot t}{1 - \cfrac{(x_\mathrm{e} + 2u_\mathrm{e})\,(y_\mathrm{o} + 2v_\mathrm{o}) \cdot t}{1 - \cfrac{(x_\mathrm{o} + 3u_\mathrm{o})\,(y_\mathrm{e} + 3v_\mathrm{e}) \cdot t}{\ddots}}}}}
$$

Separate by parity

$$P_{2n}(x_{\rm e}, x_{\rm o}, y_{\rm e}, y_{\rm o}, u_{\rm e}, u_{\rm o}, v_{\rm e}, v_{\rm o}) = \sum_{\sigma \in \mathfrak{S}_{2n}^{\rm ca}} x_{\rm e}^{\rm eareccpeakeven(\sigma)} x_{\rm o}^{\rm eareccpeakodd(\sigma)} y_{\rm e}^{\rm ereccvaleven(\sigma)} y_{\rm o}^{\rm ereccvalodd(\sigma)} \times u_{\rm e}^{\rm nrcpeakeven(\sigma)} u_{\rm o}^{\rm nrcpeakodd(\sigma)} v_{\rm e}^{\rm nrcvaleven(\sigma)} v_{\rm o}^{\rm nrcvalodd(\sigma)}$$

Theorem (D.–Sokal '23 (arxiv))

$$\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{x_{\rm e}\, y_{\rm o} \cdot t}{1 - \cfrac{(x_{\rm o} + u_{\rm o})\,(y_{\rm e} + v_{\rm e}) \cdot t}{1 - \cfrac{(x_{\rm e} + 2u_{\rm e})\,(y_{\rm o} + 2v_{\rm o}) \cdot t}{1 - \cfrac{(x_{\rm o} + 3u_{\rm o})\,(y_{\rm e} + 3v_{\rm e}) \cdot t}{\ddots}}}}}$$

Special case of more general continued fraction of Sokal–Zeng involving 2 infinite families

Counting of cycles

Consider 9-variable polynomials

$$
\begin{aligned}
& P_{2n}(x_{\mathrm{e}}, x_{\mathrm{o}}, y_{\mathrm{e}}, y_{\mathrm{o}}, u_{\mathrm{e}}, u_{\mathrm{o}}, v_{\mathrm{e}}, v_{\mathrm{o}}) \\
= & \sum_{\sigma \in \mathfrak{S}_{2n}^{\mathrm{ca}}} x_{\mathrm{e}}^{\mathrm{eareccpeakeven}(\sigma)} x_{\mathrm{o}}^{\mathrm{eareccpeakodd}(\sigma)} y_{\mathrm{e}}^{\mathrm{ereccvaleven}(\sigma)} y_{\mathrm{o}}^{\mathrm{ereccvalodd}(\sigma)} \quad \times \\
& \quad u_{\mathrm{e}}^{\mathrm{nrcpeakeven}(\sigma)} u_{\mathrm{o}}^{\mathrm{nrcpeakodd}(\sigma)} v_{\mathrm{e}}^{\mathrm{nrcvaleven}(\sigma)} v_{\mathrm{o}}^{\mathrm{nrcvalodd}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}
\end{aligned}
$$

Counting of cycles

Consider 9-variable polynomials

$$P_{2n}(x_e, x_o, y_e, y_o, u_e, u_o, v_e, v_o)$$
$$= \sum_{\sigma \in \mathfrak{S}_{2n}^{ca}} x_e^{\mathrm{eareccpeakeven}(\sigma)} x_o^{\mathrm{eareccpeakodd}(\sigma)} y_e^{\mathrm{ereccvaleven}(\sigma)} y_o^{\mathrm{ereccvalodd}(\sigma)} \times$$
$$u_e^{\mathrm{nrcpeakeven}(\sigma)} u_o^{\mathrm{nrcpeakodd}(\sigma)} v_e^{\mathrm{nrcvaleven}(\sigma)} v_o^{\mathrm{nrcvalodd}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

Need to set $y_e = v_e$, $y_o = v_o$

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Need to set $y_e = v_e$, $y_o = v_o$

Theorem (D.–Sokal '23 (arxiv))

$$\sum_{n=0}^{\infty} P_{2n} t^n = \cfrac{1}{1 - \cfrac{\lambda x_e \, y_o \cdot t}{1 - \cfrac{(\lambda+1)(x_o + u_o) y_e \cdot t}{1 - \cfrac{(\lambda+2)(x_e + 2u_e) y_o \cdot t}{1 - \cfrac{(\lambda+3)(x_o + 3u_o) y_e \cdot t}{\ddots}}}}}$$

$$\underbrace{\text{Alternating cycles}}_{E_{2n-1}} \subseteq \underbrace{\text{Cycle-alternating permutations}}_{E_{2n}}$$

Jacobian Elliptic Function

$$u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

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$$\mathrm{sn}(u, k) = \sin \mathrm{am}(u, k)$$

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Combinatorial interpretation due to Dumont (1979,1980). He introduced Schett polynomials.

Series expansion

$$\mathrm{sn}(u,k) = \sum_{n=0}^{\infty} (-1)^{(n-1)/2} \mathcal{E}_{2n+1}(k) \frac{u^{2n+1}}{(2n+1)!}$$

$$\mathrm{cn}(u,k) = \sum_{n=0}^{\infty} (-1)^{n/2} \mathcal{E}_{2n}(k) \frac{u^{2n}}{(2n)!}$$

Series expansion

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\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \mathcal{E}_{2n}(k) t^n = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{2^2k^2t}{1 - \cfrac{3^2t}{1 - \cfrac{4^2k^2t}{1 - \ddots}}}}}
$$

[Stieltjes, 1889]

Series expansion

$$\begin{aligned} \mathrm{sn}(u,k) &= \sum_{n=0}^{\infty}(-1)^{(n-1)/2}\mathcal{E}_{2n+1}(k)\frac{u^{2n+1}}{(2n+1)!} \\ \mathrm{cn}(u,k) &= \sum_{n=0}^{\infty}(-1)^{n/2}\mathcal{E}_{2n}(k)\frac{u^{2n}}{(2n)!} \end{aligned}$$

$$\sum_{n=0}^{\infty}\mathcal{E}_{2n}(k)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{2^2k^2t}{1-\cfrac{3^2t}{1-\cfrac{4^2k^2t}{1-\ddots}}}}}$$

[Stieltjes, 1889]
Our continued fraction also generalises this

Structure

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

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Define lower-triangular matrix J where

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Define lower-triangular matrix J where

$$\begin{aligned}
\mathrm{J}_{n,n} &= 1\\
\mathrm{J}_{n,k} &= \mathrm{J}_{n-1,k-1}+\gamma_k \mathrm{J}_{n-1,k}+\beta_{k+1}\mathrm{J}_{n-1,k+1}
\end{aligned}$$

Jacobi-Rogers matrix

Consider the J-fraction

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\ddots}}}$$

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Also known as Stieltjes table/tableau

If

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \ddots}}}$$

then

$$\mathrm{J}_{n,0} = a_n$$

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then

$$\mathrm{J}_{n,0} = a_n$$

Question: If J-fraction for a_n is known, combinatorially understand matrix J

When $a_n = n!$,

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - t - \cfrac{1t^2}{1 - 3t - \cfrac{4t^2}{1 - \ddots}}}$$

$$\mathrm{J}_{n,k} = \binom{n}{k}\frac{n!}{k!}$$

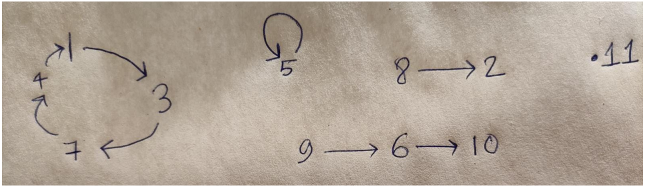
These count Laguerre digraphs with k paths

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1

Laguerre digraph

Laguerre digraph is a labelled digraph where each vertex has in and out-degree 0 or 1



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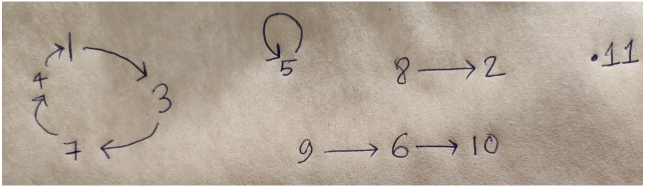

Each connected component is a cycle or a path

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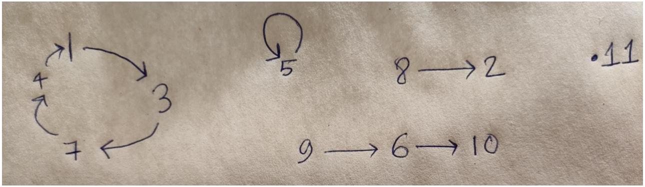

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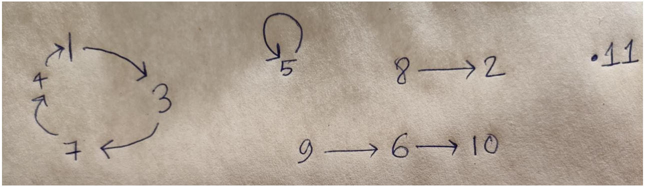

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Number of Laguerre digraphs on n vertices with k elements -

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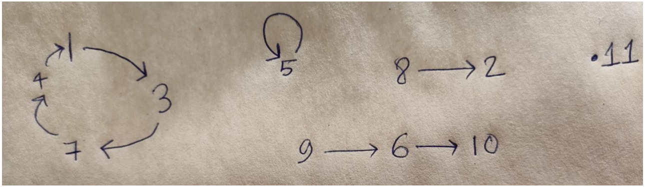

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Number of Laguerre digraphs on n vertices with k elements -

$$\mathrm{J}_{n,k} = \binom{n}{k}\frac{n!}{k!}$$

Can extend permutation statistics to Laguerre digraphs [D.–Sokal (ongoing)]

i can be classified as:

- Peak
- Valley
- Double ascent
- Double descent
- Loop

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Alternating Laguerre digraph - Laguerre digraphs where each vertex is either a peak or a valley

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Alternating Laguerre digraph - Laguerre digraphs where each vertex is either a peak or a valley

Interpret Jacobi-Rogers matrix for secant numbers E_{2n} [D.–Sokal '23]

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1-\cfrac{1\cdot t}{1-\cfrac{1\cdot t}{1-\cfrac{2\cdot t}{1-\cfrac{2\cdot t}{1-\ddots}}}}}$$

i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.

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i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.
Question: Combinatorially understand α's $1^k, 1^k, 2^k, 2^k, 3^k, 3^k, \ldots$ "multivariately"

- $k = 1$ quasi-linear case: $n!$

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- $k=1$ quasi-linear case: $n!$
- $k=2$ quasi-quadratic case: Median Genocchi numbers

Question

We have a combinatorial interpretation for

$$\cfrac{1}{1 - \cfrac{1\cdot t}{1 - \cfrac{1\cdot t}{1 - \cfrac{2\cdot t}{1 - \cfrac{2\cdot t}{1 - \ddots}}}}}$$

i.e. α's given by $1, 1, 2, 2, 3, 3, 4, 4, \ldots$. We can also read off statistics from this by putting in variables.
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- $k = 1$ quasi-linear case: $n!$
- $k = 2$ quasi-quadratic case: Median Genocchi numbers
- $k = 3$ quasi-cubic case: Not on OEIS!!!

References

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- Continued fractions for cycle-alternating permutations, B. Deb and A. Sokal, arxiv, 2023.
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Thank you