Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants

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Cycle classification

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- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

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	cpeak	cval	cdrise	cdfall	fix
erec		ereccval	ereccdrise		
earec	eareccpeak			eareccdfall	
rar					rar
nrar	nrcpeak	nrcval	nrcdrise	nrcdfall	nrfix

Continued fractions counting permutation statistics

Consider 10-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) =$$
$$\sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times$$
$$u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)}$$

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Theorem (Sokal–Zeng (2022) First J-fraction for permutations)

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) t^n$$

$$= \cfrac{1}{1 - z \cdot t - \cfrac{x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + w) \cdot t - \cfrac{(x_1 + u_1)(y_1 + v_1) \cdot t^2}{1 - ((x_2 + u_2) + (y_2 + v_2) + w) \cdot t - \cfrac{(x_1 + 2u_1)(y_1 + 2v_1) \cdot t^2}{1 - \ddots}}}}$$

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Proof uses the Foata–Zeilberger bijection (1990)

Can we count cycles as well?

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Conjecture (Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1,x_2,y_1,y_2,u_1,u_2,y_1,v_2,w,z,\lambda)t^n$$

$$= \cfrac{1}{1-\lambda z\cdot t-\cfrac{\lambda x_1\,y_1\cdot t^2}{1-(x_2+y_2+\lambda w)\cdot t-\cfrac{(\lambda+1)(x_1+u_1)y_1\cdot t^2}{1-((x_2+v_2)+(y_2+v_2)+\lambda w)\cdot t-\cfrac{(\lambda+2)(x_1+2u_1)y_1\cdot t^2}{1-\ddots}}}}$$

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Twist in story:
Can prove their full conjecture using Foata–Zeilberger bijection

We can count cycles in the Foata–Zeilberger bijection excedance indices $F \quad = \quad \{i \in \sigma : \sigma(i) > i\} \;=\;$ Cdrise ∪ Cval

$$\begin{aligned} \text{excedance indices } F &= \{i \in \sigma : \sigma(i) > i\} = \text{Cdrise} \cup \text{Cval} \\ \text{excedance values } F' &= \{i \in \sigma : i > \sigma^{-1}(i)\} = \text{Cdrise} \cup \text{Cpeak} \end{aligned}$$

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A permutation can be fully described the following data:

- Sets F, F', G, G', H

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- $\xi = (\xi_1, \dots, \xi_n)$ are labels on the steps of the Motzkin paths
  Correspond to $\sigma|_F : F \to F'$ and $\sigma|_G : G \to G'$

Description of $\sigma \to \omega$

- If i is a cycle valley, step i is $\nearrow$
- If i is a cycle peak, step i is $\searrow$
- If i is a cycle double rise, cycle double fall or fixed, step i is $\textcolor{red}{\to}$, $\textcolor{blue}{\to}$ or $\textcolor{green}{\to}$ respectively.

Description of labels $\sigma \to \xi$

For $i \in [n]$

$$\xi_i = \begin{cases} \#\{j\colon j < i \text{ and } \sigma(j) > \sigma(i)\} & \text{if } \sigma(i) > i \quad \text{if } i \in \text{ Cval} \cup \text{Cdrise} \\ \#\{j\colon j > i \text{ and } \sigma(j) < \sigma(i)\} & \text{if } \sigma(i) < i \quad \text{if } i \in \text{Cpeak} \cup \text{Cdfall} \\ 0 & \text{if } \sigma(i) = i \quad \text{if } i \in \text{ Fix} \end{cases}$$

An example

Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

- Cval $= \{1, 3\}$
- Cpeak $= \{7, 9\}$
- Cdrise $= \{5\}$
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- Fix $= \{4\}$

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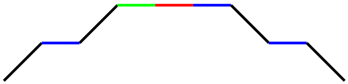

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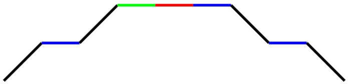

The labels ξ and the sets F, F', G, G' are:

$i \in F$	1	3	5
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Definition

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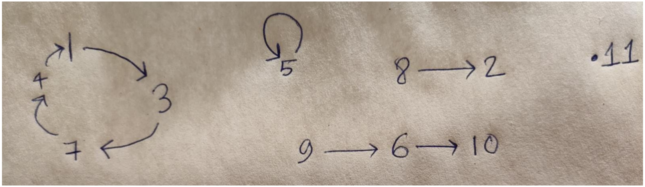

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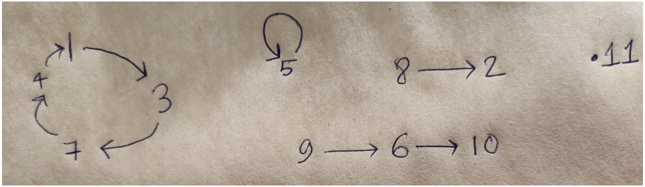

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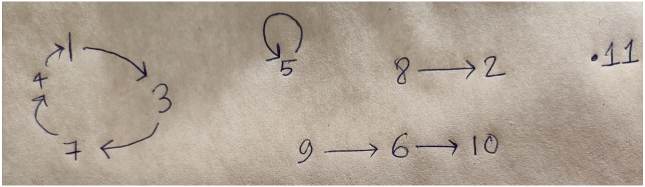

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Generalise permutations

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Stage (a): $i \in H$ (fixed points) in increasing order

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This order is suggested by the inverse bijection and the inversion tables

History with an example

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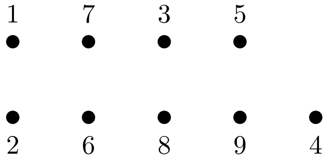

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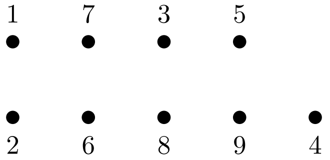

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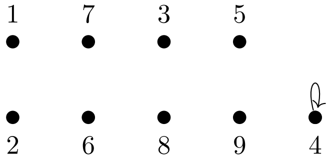

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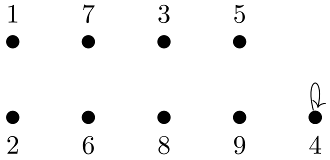
Stage (b): G in increasing order



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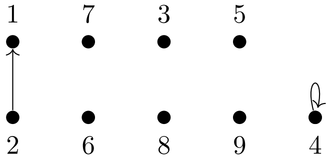
Stage (b): G in increasing order



Let $\sigma = 715492638 = (1762)(3598)(4) \in \mathfrak{S}_9$.

$H = \{4\}$

$i \in F$	1	3	5
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ξ_i	0	1	0

$i \in G$	2	6	7	8	9
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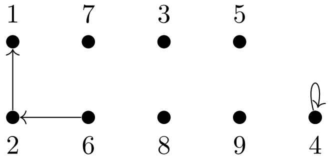

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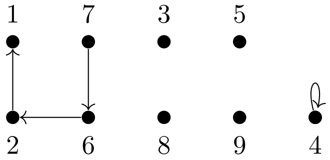

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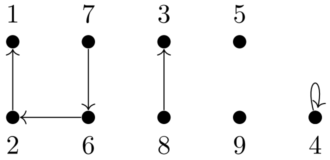

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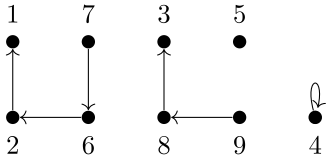

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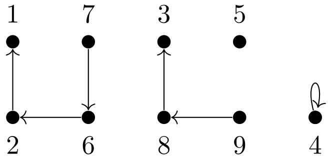
Stage (c): F in decreasing order



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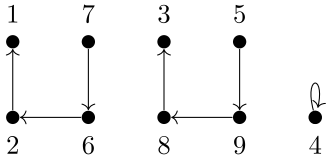

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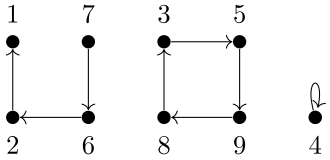

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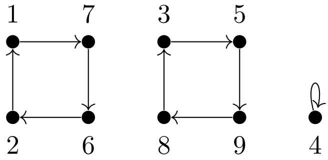

Story continues ....

This resolves the Sokal–Zeng conjecture (2022) for permutations

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Resolved a 4-variable conjectured continued fraction due to Randrianarivony–Zeng (1996) for Genocchi numbers

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Resolved a 4-variable conjectured continued fraction due to Randrianarivony–Zeng (1996) for Genocchi numbers

Similar to Sokal–Zeng, have generalised these continued fractions to families of infinitely many variables

Thank you